MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 4

- Eigenvalues and Eigenvectors.

- Diagonalization.

EIGENVALUES and EIGENVECTORS

Definition: Let A be an $n \times n$ matrix.
A vector $0 \neq x \in R^n$ is called an **eigenvector** of A if $Ax = \lambda x$ for some scalar λ. The scalar λ is called an **eigenvalue** of A.

Example: Let

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Are x and y eigenvectors of A?
Solution:

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x,$$

$$Ay = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix} \neq \mu \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

x is an eigenvector of A corresponding to an eigenvalue $\lambda = 3$, but y is not an eigenvector of A.

λ is an eigenvalue of A
$\Longleftrightarrow$ $Ax = \lambda x$ for some non-zero vector x.
$\Longleftrightarrow$ $(A - \lambda I)x = 0$ for some non-zero vector x.
$\Longleftrightarrow$ $\det(A - \lambda I) = 0$.

$\det(A - \lambda I) = 0$ is called the **characteristic equation** of A, the scalars λ satisfying this equation are the eigenvalues of A.

When expanded, $\det(A - \lambda I)$ is a polynomial in λ, it has degree n, and it is called the **characteristic polynomial** of A.

Definition: The set of all solutions of $(A-\lambda I)x = 0$ is called the **eigenspace of** A corresponding to λ, and it is denoted by E_λ.

$$\boxed{E_\lambda = \text{Nul}(A - \lambda I)}$$

Note: Eigenspace contains the zero vector. But an eigenvector is **never the zero vector**.

An eigenvalue λ might be zero:

$$\boxed{\lambda = 0 \text{ is an eigenvalue of } A \Longleftrightarrow A \text{ is not invertible.}}$$

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = 0 \Longleftrightarrow \lambda = 0, 2.$$

Thus, A is not invertible.

Example: Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)(-3-\lambda). \\ \det(A - \lambda I) &= 0 \Longleftrightarrow \lambda_1 = -3, \lambda_2 = 0, \lambda_3 = 4. \end{aligned}$$

Remark: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$.
Find the eigenvalues and the corresponding eigenvectors of A. What are the eigenspace(s)?

Solution:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{bmatrix}.$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4.$$

The characteristic equation of A is

$$\lambda^2 - 3\lambda - 4 = 0.$$

The eigenvalues of A:

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \Longleftrightarrow \lambda_1 = 4,\ \lambda_2 = -1.$$

<u>Eigenvectors for $\lambda_1 = 4$:</u>

$$A - \lambda I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}.$$

$$(A - \lambda I)x = 0 \Longleftrightarrow \left[\begin{array}{rr|r} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rr|r} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow x = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \in R.$$

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad E_4 = \text{Span}\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}.$$

A basis for E_4: $\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$, $\quad \dim E_4 = 1.$

<u>Eigenvectors for $\lambda_2 = -1$:</u>

$$A - (-1)I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$(A + I)x = 0 \Longleftrightarrow x = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

A basis for E_{-1} : $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\quad \dim E_{-1} = 1.$

Theorem: Let A be an $n \times n$ matrix.
If $v_1, v_2, ..., v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$, then $\{v_1, v_2, ..., v_r\}$ is a linearly independent set.

Example: For $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ we know that:

$$\lambda_1 = 4 \Longrightarrow v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \Longrightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

By the above theorem, $\{v_1,\ v_2\}$ is a linearly independent set.

Example: Let A and B be two $n \times n$ matrices such that

$$B = PAP^{-1}$$

for an invertible matrix P.
(In this case, we say that A is similar to B.)
Show that A and B have the same eigenvalues.
Solution:

$$B - \lambda I = PAP^{-1} - \lambda PP^{-1} = P(A - \lambda I)P^{-1}.$$

$$\begin{aligned} \det(B - \lambda I) &= \det\ \left(P(A - \lambda I)P^{-1}\right) \\ &= \det P \cdot \det\ (A - \lambda I) \cdot \det\ P^{-1} \\ &= \det P \cdot \det\ (A - \lambda I) \cdot \frac{1}{\det\ P} \\ &= \det\ (A - \lambda I). \end{aligned}$$

DIAGONALIZATION

Definition: An $n \times n$ matrix A is said to be diagonalizable if A is similar to a diagonal matrix D, i.e, if $A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D.

$$A = PDP^{-1} \Longleftrightarrow P^{-1}AP = D.$$

Theorem:

- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- An $n \times n$ matrix A is diagonalizable if and only if R^n has a basis consisting of eigenvectors of A.
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Example: For $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ we know that:

$$\lambda_1 = 4 \Longrightarrow v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \Longrightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For the matrices P and D

$$P = [v_1 \;\; v_2] = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}; \quad A = PDP^{-1}.$$

Instead of verifying the equation $A = PDP^{-1}$, we can verify $AP = PD$.

Example: Compute A^{10}, where $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$.

Solution:

$A = PDP^{-1} \Longleftrightarrow A^n = PD^nP^{-1}$.

$$\begin{aligned}
A &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\
A^{10} &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{10} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1048576 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}
\end{aligned}$$

$$= \frac{1}{5}\begin{bmatrix} 4194304 & -1 \\ 1048576 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 838861 & 838860 \\ 209715 & 209716 \end{bmatrix}.$$

Example: Is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable?
Solution: Since A is upper triangular, the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$.

$$A - \lambda_i I = A \quad (i = 1, 2) \quad \text{and} \quad Ax = 0 \Longleftrightarrow x = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}, t \in R.$$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 0$.

A is a 2×2 matrix but it does not have 2 linearly independent eigenvectors. Hence, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.
Solution: Since A is a triangular matrix, eigenvalues of A are the entries on the main diagonal, i.e, $\lambda_1 = 1$, $\lambda_2 = -4$, and $\lambda_3 = -2$.
Eigenvectors corresponding to $\lambda_1 = 1$:

$$A - \lambda_1 I \;=\; A - I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -5 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_1 \in R.$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 1$.

Eigenvectors corresponding to $\lambda_2 = -4$:

$$A - \lambda_2 I = A - (-4)I = A + 4I = \begin{bmatrix} 5 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 4I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/5)x_2 \\ x_2 \\ 0 \end{bmatrix} = (1/5)x_2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

$v_2 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -4$.

Eigenvectors corresponding to $\lambda_3 = -2$:

$$A - \lambda_3 I = A - (-2)I = A + 2I = \begin{bmatrix} 3 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+2I)X=0 \Longleftrightarrow X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} (1/3)x_3 \\ x_3 \\ x_3 \end{bmatrix}=(1/3)x_3\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}=t\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

$v_3=\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda_3=-2$.
Since A is a 3×3 matrix and it has three linearly independent eigenvectors,

$$v_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ v_2=\begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix},\ v_3=\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

A is diagonalizable.

Let

$$D=\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ \text{and}$$

$$P=[v_1\ \ v_2\ \ v_3]=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}.\ \text{Then,}$$

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1/5 & -2/15 \\ 0 & 1/5 & -1/5 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$.

a) Find the eigenvalues and corresponding eigenvectors of A.
b) Is A diagonalizable? If yes, write A as $A = PDP^{-1}$, where D is a diagonal matrix.

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} \\ &= -(1+\lambda)(\lambda^2 - 6\lambda + 5) \\ &= -(1+\lambda)(\lambda - 5)(\lambda - 1) = 0 \Longleftrightarrow \lambda = -1,\ 5,\ 1. \end{aligned}$$

$\underline{\lambda_1 = -1:}$

$$A - \lambda I = A + I = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$E_{-1} = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, a basis for E_{-1} is $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$, $\dim E_{-1} = 1$.

$\underline{\lambda_2 = 5:}$

$$A - \lambda I = A - 5I = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 5I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$E_5 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$, a basis for E_5 is $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$, $\dim E_5 = 1$

$\underline{\lambda_3 = 1:}$

$$A - \lambda I \;=\; A - I = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \ x_3 \in R.$$

$E_1 = \text{Span}\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$, a basis for E_1 is $\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$, $\dim E_1 = 1$.

A is a 3×3 matrix and the eigenvectors of A

$$\left\{ v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

form a basis for R^3.
Thus A is diagonalizable, and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Find the eigenvalues and corresponding eigenvectors of A.
b) Diagonalizable A, if possible.
Solution:

$$\det(A - \lambda I) = -(\lambda - 5)^2(\lambda - 1) = 0 \iff \lambda_1 = 5,\ \lambda_2 = 1.$$

$\underline{\lambda_1 = 5:}$

$$A - 5I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 5I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_1 = 5 : \qquad v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_5 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \dim E_5 = 2.$$

$\underline{\lambda_2 = 1:}$

$$A - I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \iff x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 \in R.$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \dim E_1 = 1.$$

Thus $A = PDP^{-1}$, where

$$P = [v_1\ v_2\ v_3] = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Even though A is a 3×3 matrix and it has only two distinct eigenvalues, its eigenvectors

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

form a basis for R^3.

$$P_1 = \begin{bmatrix} v_2 & v_1 & v_3 \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \Longrightarrow A = P_1 D P_1^{-1}.$$

$$P' = \begin{bmatrix} v_1 & v_3 & v_2 \end{bmatrix}, D' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \Longrightarrow A = P' D' P'^{-1}.$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$. Is A diagonalizable?

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & -1-\lambda \\ -1 & 1 \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 + 3\lambda) - 3\lambda \\ &= -\lambda^2(\lambda + 2) = 0 \Longleftrightarrow \lambda_1 = 0, \lambda_2 = -2. \end{aligned}$$

$\underline{\lambda_1 = 0:}$

$$A - \lambda I = A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix},\ x_3 \in R.$$

$$v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \quad E_0 = \text{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\underline{\lambda_2 = -2:}$

$$A - (-2)I = A + 2I = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},\ x_3 \in R.$$

$$v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad E_{-2} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

A is a 3×3 matrix but it has only two linearly independent eigenvectors:

$$\left\{ v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\},$$

which is not a basis for R^3. Hence, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. Diagonalize A if possible.

Solution:

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 2 & 0 & 0 & 1-\lambda \end{vmatrix} \\
&= (4-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} \\
&= (4-\lambda)(3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\
&= (4-\lambda)(3-\lambda)(\lambda^2-2\lambda-3) \\
&= (4-\lambda)(3-\lambda)(\lambda-3)(\lambda+1) = 0 \Longleftrightarrow \lambda = -1, 3, 4.
\end{aligned}$$

$\underline{\lambda_1 = -1:}$

$$A + I = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A+I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span}\{v_1\}, \quad \dim E_{-1} = 1.$$

$\underline{\lambda_2 = 3:}$

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 3I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_3 = \mathrm{Span}\{v_2, v_3\}, \quad \mathrm{dim} E_3 = 2.$$

$\underline{\lambda_3 = 4:}$

$$A - 4I \;\; = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 4I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \in R.$$

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; E_4 = \mathrm{Span}\{v_4\}, \; \mathrm{dim} E_4 = 1.$$

A is a 4×4 matrix and

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

a basis for R^4. So A is diagonalizable.

$$P = [v_1 \;\; v_2 \;\; v_3 \;\; v_4] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix};$$

$$A = PDP^{-1}$$